CS 0368-4246: Combinatorial Methods in Algorithms (Spring 2025) June 9, 2025

Lecture 10: Spectral Expanders

Instructor: *Or Zamir*

Homework Questions

1. Let G be a graph and $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ the sorted eigenvalues of its normalized Laplacian N_G. Prove that $\lambda_k = 0$ if and only if the number of connected components of G is at least k.

2. (The Expander Mixing Lemma) Let G be a d-regular graph with n vertices, and denote by λ the second largest eigenvalue of A_G in absolute value[1]. Let $S, T \subseteq V(G)$ be vertex sets, prove that

$$\left| e(S,T) - \frac{d}{n}|S||T| \right| \leq \lambda \sqrt{|S||T|}.$$

[1] As we now discuss regular graphs, we can talk directly about the eigenvalues of A_G rather than the normalized version N_G which in this case is $N_G = I - \frac{1}{d}A_G$. Note that x is an eigenvalue of A_G if and only if $1 - x/d$ is an eigenvalue of N_G. Thus, the largest eigenvalue of A_G is always d, and the expansion is determined by the *gap* between the largest and second largest eigenvalues.