CS 0368-4246: Combinatorial Methods in Algorithms (Spring 2025) April 28, 2025

Lecture 6: Edge Contractions in Graph Algorithms

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Homework Questions

1. In class we proved that any graph G has at most $\binom{n}{2}$ minimum cuts. Prove that this is tight: Construct a graph family in which the number of minimum cuts is exactly $\binom{n}{2}$ (or at least $\Omega(n^2)$).
2. Analysis of "Galton-Watson" branching processes: Let k, p be two fixed parameters. Denote by T the complete k-ary tree of height h in which we remove each vertex (or equivalently, edge) independently with probability p. Denote by p_h the probability that at least one root-to-leaf path "survived" the removals.
   (a) If $p > \frac{1}{k}$, prove that there exists a constant $\alpha > 0$ such that $p_h \geq \alpha$ for all $h \in \mathbb{N}$.
   (b) If $p = \frac{1}{k}$, prove that $p_h = \Omega(\frac{1}{h})$.
   (c) If $p < \frac{1}{k}$, prove that there exists a constant $\beta \in (0, 1)$ such that $p_h = O(\beta^h)$.
3. We said that H is a **minor** of G if it can be reached from G by a sequence of edge or vertex removals and edge contractions. Let H be a graph of fixed size $r := |V(H)|$. Prove that every H-minor-free graph has at most $2^{r-1} \cdot n$ edges. In particular, every graph that avoids a fixed size minor must be sparse. (Hint: prove by induction on the number of vertices n).