CS 0368-4246: Combinatorial Methods in Algorithms (Spring 2025) July 23, 2025

Final Exam (A)

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Instructions (Please Read Carefully)

- This exam consists of **four** questions. The number of points for each question and sub-question is indicated alongside it. The total available points add up to **105**, but the final score will be capped at **100**.
- You may use theorems or lemmas that were proved in class, provided you cite them exactly as stated. Any modifications of these results—or use of results from homework—must be fully re-proven.
- Write your answers clearly and in an organized manner in the provided exam booklet. Clearly label each answer with the corresponding question number, and indicate whether each page is a draft or a final answer.
- Unless explicitly stated otherwise, all graphs are assumed to be undirected and unweighted, and all notation follows that used in class.
- Unless explicitly stated otherwise, algorithms with an expected run-time or that succeed with high probability are sufficient.

Exam Questions

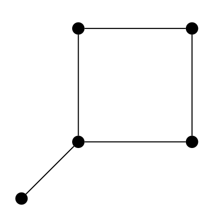

Figure 1: The kite graph $\mathcal{K}$.

1. *(40 points)* Denote by $\mathcal{K}$ a graph on five vertices we would call *a kite*, this is a 4-cycle C_4 with an additional node connected to exactly one of the cycle vertices, see Figure 1.

   (a) *(15 points)* Show that given a graph G with n vertices, we can check if it contains a kite $\mathcal{K}$ as a subgraph in $\tilde{O}(n^2)$ time.

   (b) *(10 points)* Assume that there is no algorithm for C_4-detection in n-vertex graphs running in $n^{2-\varepsilon}$ time for any constant $\varepsilon > 0$. Prove that no algorithm can detect a kite in n-vertex graphs in time $n^{2-\varepsilon}$ for any constant $\varepsilon > 0$.

   (c) *(15 points)* What is the extremal number of $\mathcal{K}$? Give tight, up to constant factors, upper and lower bounds for $ex(\mathcal{K}, n)$.

2. *(40 points)*

   (a) *(15 points)* Let G be a connected graph and φ a parameter. Assume that for any subset of edges $D \subseteq E(G)$ if we denote by $C_1, \ldots, C_r$ the connected components of $G - D$ for which $vol(C_i) \leq \frac{1}{2} vol(V(G))$, it holds that $\sum_{i=1}^{r} vol(C_i) \leq |D|/\varphi$. Prove that the conductance of G is at least $\Omega(\varphi)$.

   (b) *(15 points)* Let G be a graph without isolated vertices, prove that $\lambda_2(N_G) > 0$ if and only if G is connected. Note that we proved the same statement in class for the Laplacian L_G rather than for the normalized Laplacian $N_G := D_G^{-1/2} L_G D_G^{-1/2} = I - D_G^{-1/2} A_G D_G^{-1/2}$.

   (c) *(10 points)* Strengthen the previous statement and show that for every connected G with n vertices we have $\lambda_2(N_G) \geq \Omega(1/n^4)$. (Note: this is not necessarily a tight bound).

3. *(15 points)* Let G be a *weighted* graph. A **MinCut query** is an algorithm that, given a subset $S \subseteq V$, returns the minimum-weight edge with one endpoint in S and one in $V \setminus S$, if such an edge exists. You are given the vertex set of G, but have no other access to the graph (in particular, you cannot check whether an edge exists, or query degrees or weights). Your access is only through MinCut queries. Design an algorithm that computes a minimum spanning tree of G using as few MinCut queries as possible. Provide a full analysis, but there is no need to prove your solution is optimal.

4. *(10 points)* Let G be graph with n vertices. Give a $O^\star(2^n)$ time algorithm that finds the smallest number of cliques (of any sizes) needed to cover all vertices of G (this is a collection of cliques in G such that the union of their vertex sets is V).